

# A SHARP BOUNDEDNESS RESULT CONCERNING SOME MAXIMAL OPERATORS OF VILENKIN-FEJÉR MEANS

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**ABSTRACT.** In this paper we derive the maximal subspace of positive numbers, for which the restricted maximal operator of Fejér means in this subspace is bounded from the Hardy space  $H_p$  to the space  $L_p$  for all  $0 < p \leq 1/2$ . Moreover, we prove that the result is in a sense sharp.

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## 1. INTRODUCTION

In the one-dimensional case the weak (1,1)-type inequality for the maximal operator of Fejér means

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|$$

can be found in Schipp [11] for Walsh series and in Pál, Simon [10] for bounded Vilenkin series. Fujji [5] and Simon [13] verified that  $\sigma^*$  is bounded from  $H_1$  to  $L_1$ . Weisz [20] generalized this result and proved boundedness of  $\sigma^*$  from the martingale space  $H_p$  to the Lebesgue space  $L_p$  for  $p > 1/2$ . Simon [12] gave a counterexample, which shows that boundedness does not hold for  $0 < p < 1/2$ . A counterexample for  $p = 1/2$  was given by Goginava [8] (see also [2] and [3]). Weisz [21] proved that the maximal operator of the Fejér means  $\sigma^*$  is bounded from the Hardy space  $H_{1/2}$  to the space  $weak-L_{1/2}$ . The boundedness of weighted maximal operators are considered in [7], [9], [15], [16]. Weisz [19] (see also [18]) also proved that the following theorem is true:

**Theorem W: (Weisz)** Let  $p > 0$ . Then the maximal operator

$$\sigma^{\nabla,*} f = \sup_{n \in \mathbb{N}} |\sigma_{M_n} f|$$

is bounded from the Hardy space  $H_p$  to the space  $L_p$ .

The main aim of this paper is to generalize Theorem W and find the maximal subspace of positive numbers, for which the restricted maximal operator of Fejér means in this subspace is bounded from the Hardy space

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$H_p$  to the space  $L_p$  for all  $0 < p \leq 1/2$ . As applications, both some well-known and new results are pointed out.

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary Lemmas, some of them are new and of independent interest. These results are presented in Section 4. The detailed proofs are given in Section 5.

## 2. DEFINITIONS AND NOTATIONS

Denote by  $\mathbb{N}_+$  the set of the positive integers,  $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ . Let  $m := (m_0, m_1, \dots)$  be a sequence of the positive integers not less than 2. Denote by  $Z_{m_n} := \{0, 1, \dots, m_n - 1\}$  the additive group of integers modulo  $m_n$ . Define the group  $G_m$  as the complete direct product of the groups  $Z_{m_n}$  with the product of the discrete topologies of  $Z_{m_n}$ 's. In this paper we discuss bounded Vilenkin groups, i.e. the case when  $\sup_{n \in \mathbb{N}} m_n < \infty$ .

The direct product  $\mu$  of the measures  $\mu_n(\{j\}) := 1/m_n$ , ( $j \in Z_{m_n}$ ) is the Haar measure on  $G_m$  with  $\mu(G_m) = 1$ .

The elements of  $G_m$  are represented by sequences

$$x := (x_0, x_1, \dots, x_n, \dots), \quad (x_n \in Z_{m_n}).$$

It is easy to give a base for the neighbourhood of  $G_m$  :

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, \quad n \in \mathbb{N}).$$

Set  $I_n := I_n(0)$ , for  $n \in \mathbb{N}_+$  and

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m \quad (n \in \mathbb{N}).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1}, \dots, x_{N-1}), & k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0), & l = N. \end{cases}$$

It is easy to show that

$$(1) \quad \overline{I_N} = \left( \bigcup_{i=0}^{N-2} \bigcup_{j=i+1}^{N-1} I_N^{i,j} \right) \cup \left( \bigcup_{i=0}^{N-1} I_N^{i,N} \right).$$

If we define the so-called generalized number system based on  $m$  in the following way :

$$M_0 := 1, \quad M_{n+1} := m_n M_n \quad (n \in \mathbb{N}),$$

then every  $n \in \mathbb{N}$  can be uniquely expressed as  $n = \sum_{k=0}^{\infty} n_k M_k$ , where  $n_k \in Z_{m_k}$  ( $k \in \mathbb{N}_+$ ) and only a finite number of  $n_k$ 's differ from zero. Let

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\} \quad \text{and} \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},$$

that is  $M_{|n|} \leq n \leq M_{|n|+1}$ . Set  $d(n) = |n| - \langle n \rangle$ , for all  $n \in \mathbb{N}$ .

Next, we introduce on  $G_m$  an orthonormal system, which is called the Vilenkin system. At first, we define the complex-valued function  $r_k(x) : G_m \rightarrow \mathbb{C}$ , the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k), \quad (i^2 = -1, x \in G_m, k \in \mathbb{N}).$$

Now, define the Vilenkin system  $\psi := (\psi_n : n \in \mathbb{N})$  on  $G_m$  as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system, when  $m \equiv 2$ .

The norms (or quasi-norms) of the spaces  $L_p(G_m)$  and *weak* -  $L_p(G_m)$  ( $0 < p < \infty$ ) are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p d\mu, \quad \|f\|_{\text{weak-L}_p}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < \infty.$$

The Vilenkin system is orthonormal and complete in  $L_2(G_m)$  (see [17]).

If  $f \in L_1(G_m)$  we can define Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi_k} d\mu \quad (k \in \mathbb{N}),$$

$$\begin{aligned} S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, & D_n &:= \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+), \\ \sigma_n f &:= \frac{1}{n} \sum_{k=0}^{n-1} S_k f, & K_n &:= \frac{1}{n} \sum_{k=0}^{n-1} D_k \quad (n \in \mathbb{N}_+). \end{aligned}$$

Recall that (see e.g. [1])

$$(2) \quad D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases}$$

and

$$(3) \quad D_{s_n M_n} = D_{s_n M_n} \sum_{k=0}^{s_n-1} \psi_k M_n = D_{M_n} \sum_{k=0}^{s_n-1} r_n^k,$$

where  $n \in \mathbb{N}$  and  $1 \leq s_n \leq m_n - 1$ .

The  $\sigma$ -algebra generated by the intervals  $\{I_n(x) : x \in G_m\}$  will be denoted by  $F_n$  ( $n \in \mathbb{N}$ ). Denote by  $f = (f^{(n)}, n \in \mathbb{N})$  a martingale with respect to  $F_n$  ( $n \in \mathbb{N}$ ) (for details see e.g. [18]). The maximal function of a martingale  $f$  is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

In the case  $f \in L_1(G_m)$ , the maximal functions are also be given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.$$

For  $0 < p < \infty$  the Hardy martingale spaces  $H_p(G_m)$  consist of all martingales  $f$ , for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If  $f \in L_1(G_m)$ , then it is easy to show that the sequence  $(S_{M_n}(f) : n \in \mathbb{N})$  is a martingale. If  $f = (f^{(n)}, n \in \mathbb{N})$  is martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f^{(k)}(x) \overline{\psi_i}(x) d\mu(x).$$

The Vilenkin-Fourier coefficients of  $f \in L_1(G_m)$  are the same as those of the martingale  $(S_{M_n}(f) : n \in \mathbb{N})$  obtained from  $f$ .

A bounded measurable function  $a$  is said to be a  $p$ -atom if there exists an interval  $I$ , such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

### 3. THE MAIN RESULT AND APPLICATIONS

Our main result reads:

**Theorem 1.** *a) Let  $0 < p \leq 1/2$  and  $\{n_k : k \geq 0\}$  be a sequence of positive numbers, such that*

$$\sup_k \rho(n_k) \leq c < \infty.$$

*Then the maximal operator*

$$\widetilde{\sigma}^{*, \nabla} f = \sup_{k \in \mathbb{N}} |\sigma_{n_k} f|$$

*is bounded from the Hardy space  $H_p$  to the space  $L_p$ .*

*The statement in a) is sharp in the following sense:*

*b) Let  $0 < p < 1/2$  and  $\{n_k : k \geq 0\}$  be a sequence of positive numbers, such that*

$$(4) \quad \sup_k \rho(n_k) = \infty.$$

*Then there exists a martingale  $f \in H_p$  such that*

$$\sup_{k \in \mathbb{N}} \|\sigma_{n_k} f\|_p = \infty.$$

As a first application we obtain the previous mentioned result by Weisz [18], [19] (Theorem W).

**Corollary 1.** *Let  $p > 0$  and  $f \in H_p$ . Then the maximal operator  $\sigma^{\nabla, *} f$  is bounded from the Hardy space  $H_p$  to the space  $L_p$ .*

Moreover, we get the following new information:

**Corollary 2.** *Let  $0 < p < 1/2$ ,  $f \in H_p$  and  $\{n_k : k \geq 0\}$  be any sequence of positive numbers. Then the maximal operator*

$$\tilde{\sigma}^{*, \nabla} f = \sup_{k \in \mathbb{N}} |\sigma_{n_k} f|$$

*is bounded from the Hardy space  $H_p$  to the space  $L_p$  if and only if*

$$\sup_k \rho(n_k) < \infty.$$

**Corollary 3.** *Let  $0 < p < 1/2$ ,  $f \in H_p$  and  $\{n_k : k \geq 0\}$  be any sequence of positive numbers. Then  $\sigma_{n_k} f$  are uniformly bounded from the Hardy space  $H_p$  to the space  $L_p$  if and only if*

$$\sup_k \rho(n_k) < \infty.$$

#### 4. AUXILIARY LEMMAS

For the proof of Theorem 1 we need the following Lemmas:

**Lemma 1** (see e.g. [19]). *A martingale  $f = (f^{(n)}, n \in \mathbb{N})$  is in  $H_p$  ( $0 < p \leq 1$ ) if and only if there exist a sequence  $(a_k, k \in \mathbb{N})$  of  $p$ -atoms and a sequence  $(\mu_k, k \in \mathbb{N})$  of real numbers such that for every  $n \in \mathbb{N}$ :*

$$(5) \quad \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}$$

and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,  $\|f\|_{H_p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p}$ , where the infimum is taken over all decomposition of  $f$  of the form (5).

**Lemma 2** (see e.g. [19]). *Suppose that an operator  $T$  is  $\sigma$ -linear and for some  $0 < p \leq 1$*

$$\int_{\bar{I}} |Ta|^p d\mu \leq c_p < \infty,$$

*for every  $p$ -atom  $a$ , where  $I$  denote the support of the atom. If  $T$  is bounded from  $L_{\infty}$  to  $L_{\infty}$ , then*

$$\|Tf\|_p \leq c_p \|f\|_{H_p}.$$

**Lemma 3** (see [6]). *Let  $n > t$ ,  $t, n \in \mathbb{N}$ ,  $x \in I_t \setminus I_{t+1}$ . Then*

$$K_{M_n}(x) = \begin{cases} 0, & \text{if } x - x_t e_t \notin I_n, \\ \frac{M_t}{1-r_t(x)}, & \text{if } x - x_t e_t \in I_n. \end{cases}$$

**Lemma 4** (see [16]). *Let  $x \in I_N^{i,j}$ ,  $i = 0, \dots, N-1$ ,  $j = i+1, \dots, N$ . Then*

$$\int_{I_N} |K_n(x-t)| d\mu(t) \leq \frac{cM_i M_j}{M_N^2}.$$

We also need the next two new Lemmas of independent interest:

**Lemma 5.** *Let  $t, s_n, n \in \mathbb{N}$ , and  $1 \leq s_n \leq m_n - 1$ . Then*

$$s_n M_n K_{s_n M_n} = \sum_{l=0}^{s_n-1} \left( \sum_{i=0}^{l-1} r_n^i \right) M_n D_{M_n} + \left( \sum_{l=0}^{s_n-1} r_n^l \right) M_n K_{M_n}$$

and

$$|K_{s_n M_n}(x)| \geq \frac{M_n}{\sqrt{2\pi s_n}}, \text{ for } x \in I_{n+1}(e_{n-1} + e_n).$$

Moreover, if  $x \in I_t/I_{t+1}$ ,  $x - x_t e_t \notin I_n$  and  $n > t$ , then

$$(6) \quad K_{s_n M_n}(x) = 0.$$

*Proof.* We can write that

$$\begin{aligned} (7) \quad s_n M_n K_{s_n M_n} &= \sum_{l=0}^{s_n-1} \sum_{k=lM_n}^{(l+1)M_n-1} D_k \\ &= \sum_{l=0}^{s_n-1} \sum_{k=lM_n}^{(l+1)M_n-1} D_k = \sum_{l=0}^{s_n-1} \sum_{k=0}^{M_n-1} D_{k+lM_n}. \end{aligned}$$

Let  $0 \leq k < M_n$ . Since

$$(8) \quad D_{j+lM_n} = D_{lM_n} + \psi_{M_n}^l D_j = D_{lM_n} + r_n^l D_j, \text{ when } j < lM_n$$

if we apply (3) we obtain that

$$\begin{aligned} D_{k+lM_n} &= \sum_{m=0}^{lM_n-1} \psi_m + \sum_{m=lM_n}^{lM_n+k-1} \psi_m \\ &= D_{lM_n} + \sum_{m=0}^{k-1} \psi_{m+lM_n} = D_{lM_n} + r_n^l \sum_{m=0}^{k-1} \psi_m \\ &= \left( \sum_{s=0}^{l-1} r_n^s \right) D_{M_n} + r_n^l D_k. \end{aligned}$$

By applying (7) we get that

$$\begin{aligned}
s_n M_n K_{s_n M_n} &= \sum_{l=0}^{s_n-1} \sum_{k=0}^{M_n-1} D_{k+lM_n} \\
&= \sum_{l=0}^{s_n-1} \sum_{k=0}^{M_n-1} \left( \left( \sum_{i=0}^{l-1} r_n^i \right) D_{M_n} + r_n^l D_k \right) \\
&= \sum_{l=0}^{s_n-1} \left( \sum_{i=0}^{l-1} r_n^i \right) M_n D_{M_n} + \sum_{l=0}^{s_n-1} r_n^l \sum_{k=0}^{M_n-1} D_k \\
&= \sum_{l=0}^{s_n-1} \left( \sum_{i=0}^{l-1} r_n^i \right) M_n D_{M_n} + \sum_{l=0}^{s_n-1} r_n^l M_n K_{M_n}.
\end{aligned}$$

Let  $x \in I_{n+1} (e_{n-1} + e_n)$ . By Lemma 3 we have that

$$(9) \quad |K_{M_n}(x)| = \frac{M_{n-1}}{|1 - r_{n-1}(x)|} = \frac{M_{n-1}}{\sqrt{2} \sin \pi / m_{n-1}}.$$

Moreover, since

$$\begin{aligned}
\sum_{u=1}^{s_n-1} r_n^u(x) &= \sum_{u=1}^{s_n-1} \cos u + \sum_{u=1}^{s_n-1} \sin u \\
&= \frac{\cos \pi s_n / m_n \sin \pi (s_n - 1) / m_n}{\sin \pi / m_n} i + \frac{\sin \pi s_n / m_n \sin \pi (s_n - 1) / m_n}{\sin \pi / m_n},
\end{aligned}$$

it follows that

$$(10) \quad \left| \sum_{u=1}^{s_n-1} r_n^u(x) \right| = \frac{\sin \pi (s_n - 1) / m_n}{\sin \pi / m_n} \geq 1.$$

By combining (2), (10) and the first part of Lemma 5 we immediately get that

$$\begin{aligned}
|s_n M_n K_{s_n M_n}(x)| &= \left| \left( \sum_{l=0}^{s_n-1} r_n^l(x) \right) M_n K_{M_n}(x) \right| \\
&= \frac{M_n M_{n-1}}{\sqrt{2} \sin \pi / m_{n-1}} \geq \frac{M_n M_{n-1} m_{n-1}}{\sqrt{2} \pi} \geq \frac{M_n^2}{\sqrt{2} \pi}.
\end{aligned}$$

Now, let  $t, s_n, n \in \mathbb{N}$ ,  $n > t$ ,  $x \in I_t \setminus I_{t+1}$ . If  $x - x_t e_t \notin I_n$ , then, by combining (2), (3) and Lemma 3, we obtain that

$$D_{M_n}(x) = K_{M_n}(x) = 0.$$

By again using the first part of Lemma 5 we get that

$$K_{s_n M_n}(x) = 0.$$

The proof is complete.  $\square$

**Lemma 6.** *Let  $n \in \mathbb{N}$ . Then*

$$(11) \quad |K_n(x)| \leq \frac{c}{n} \sum_{l=\langle n \rangle}^{|n|} M_l |K_{M_l}| \leq c \sum_{l=\langle n \rangle}^{|n|} |K_{M_l}|$$

and

$$(12) \quad |nK_n| \geq \frac{M_{\langle n \rangle}^2}{\sqrt{2\pi\lambda}}, \quad x \in I_{\langle n \rangle+1}(e_{\langle n \rangle-1} + e_{\langle n \rangle}),$$

where  $\lambda := \sup m_n$ .

*Proof.* It is well-known that (see [4])

$$\begin{aligned} nK_n &= \sum_{k=1}^r \left( \prod_{j=1}^{k-1} r_{n_j}^{s_j} \right) s_k M_{n_k} K_{s_k M_{n_k}} \\ &\quad + \sum_{k=1}^{r-1} \left( \prod_{j=1}^{k-1} r_{n_j}^{s_j} \right) n^{(k)} D_{s_k M_{n_k}}. \end{aligned}$$

Hence the proof follows by just combining (2) and (3) with Lemmas 3 and 5.  $\square$

## 5. PROOF OF THEOREM 1

*Proof of Theorem 1.* Since

$$(13) \quad \sup_n \int_{G_m} |K_n(x)| d\mu(x) \leq c < \infty$$

we obtain that  $\tilde{\sigma}^{*,\Delta}$  is bounded from  $L_\infty$  to  $L_\infty$ . According to Lemma 2 we find that the proof of Theorem 1 will be complete, if we show that

$$\int_{I_N} |\tilde{\sigma}^{*,\Delta} a(x)| < c < \infty,$$

for every  $p$ -atom  $a$ , with support  $I$  and  $\mu(I) = M_N^{-1}$ . We may assume that  $I = I_N$ . It is easy to see that  $\sigma_{n_k}(a) = 0$  when  $n_k \leq M_N$ . Therefore, we can suppose that  $n_k > M_N$ .

Since  $\|a\|_\infty \leq M_N^{1/p}$  we find that

$$\begin{aligned} (14) \quad |\sigma_{n_k} a(x)| &\leq \int_{I_N} |a(t)| |K_{n_k}(x-t)| d\mu(t) \\ &\leq \|a\|_\infty \int_{I_N} |K_{n_k}(x-t)| d\mu(t) \leq M_N^{1/p} \int_{I_N} |K_{n_k}(x-t)| d\mu(t). \end{aligned}$$

Without lost the generality we may assume that  $i < j$ . Let  $x \in I_N^{i,j}$  and  $j < \langle n_k \rangle$ . Then  $x-t \in I_N^{i,j}$  for  $t \in I_N$  and according to Lemma 3, we obtain that

$$|K_{M_l}(x-t)| = 0, \quad \text{for all } \langle n_k \rangle \leq l \leq |n_k|.$$



By applying (14) and (11) in Lemma 6, we get that

$$(15) \quad |\sigma_{n_k} a(x)| \leq M_N^{1/p} \sum_{l=\langle n_k \rangle}^{|n_k|} \int_{I_N} |K_{M_l}(x-t)| d\mu(t) = 0,$$

$$\text{for } x \in I_N^{i,j}, \quad 0 \leq i < j < \langle n_k \rangle \leq l \leq |n_k|.$$

Let  $x \in I_N^{i,j}$ , where  $\langle n_k \rangle \leq j \leq N$ . Then, in the view of Lemma 4, we have that

$$\int_{I_N} |K_{n_k}(x-t)| d\mu(t) \leq \frac{cM_i M_j}{M_N^2}.$$

By using again (14) we find that

$$(16) \quad |\sigma_{n_k} a(x)| \leq c_p M_N^{1/p-2} M_i M_j.$$

Since  $n_k \geq M_N$  we obtain that  $|n_k| \geq N$  and

$$\sup_k (N - \langle n_k \rangle) \leq \sup_k (|n_k| - \langle n_k \rangle) \leq \sup_k \rho(n_k) < c < \infty.$$

Thus,

$$(17) \quad \frac{M_N^{1-p}}{M_{\langle n_k \rangle}^{1-p}} \leq \frac{M_{|n_k|}^{1-p}}{M_{\langle n_k \rangle}^{1-p}} \leq \lambda^{(|n_k| - \langle n_k \rangle)(1-p)} = \lambda^{\rho(n_k)(1-p)} < c < \infty,$$

where  $\lambda = \sup_k m_k$ .

By combining (1) and (14)-(17) we get that

$$\begin{aligned} & \int_{I_N} |\tilde{\sigma}^{*,\Delta} a|^p d\mu \\ &= \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \sum_{x_s=0, s \in \{j+1, \dots, N-1\}}^{m_s-1} \int_{I_N^{i,j}} |\tilde{\sigma}^{*,\Delta} a|^p d\mu + \sum_{i=0}^{N-1} \int_{I_N^{i,N}} |\tilde{\sigma}^{*,\Delta} a|^p d\mu \\ &\leq \sum_{i=0}^{\langle n_k \rangle-1} \sum_{j=\langle n_k \rangle}^{N-1} \sum_{x_s=0, s \in \{j+1, \dots, N-1\}}^{m_s-1} \int_{I_N^{i,j}} |\tilde{\sigma}^{*,\Delta} a|^p d\mu \\ &\quad + \sum_{i=\langle n_k \rangle}^{N-2} \sum_{j=i+1}^{N-1} \sum_{x_s=0, s \in \{j+1, \dots, N-1\}}^{m_s-1} \int_{I_N^{i,j}} |\tilde{\sigma}^{*,\Delta} a|^p d\mu + \sum_{i=0}^{N-1} \int_{I_N^{i,N}} |\tilde{\sigma}^{*,\Delta} a|^p d\mu \\ &\leq c_p M_N^{1-2p} \sum_{i=0}^{\langle n_k \rangle} M_i^p \sum_{j=\langle n_k \rangle+1}^{N-1} \frac{1}{M_j^{1-p}} + M_N^{1-2p} \sum_{i=\langle n_k \rangle}^{N-2} M_i^p \sum_{j=i+1}^{N-1} \frac{1}{M_j^{1-p}} + c_p \sum_{i=0}^{N-1} \frac{M_i^p}{M_N^p} \\ &\leq \frac{c_p M_N^{1-2p}}{M_{\langle n_k \rangle}^{1-2p}} + c_p \leq \frac{c_p M_{|n_k|}^{1-p}}{M_{\langle n_k \rangle}^{1-p}} + c_p \leq c_p \lambda^{(|n_k| - \langle n_k \rangle)(1-p)} < \infty. \end{aligned}$$

The proof of the a) part is complete.

Now, we prove the b) part of Theorem 1. Let  $\{n_k : k \geq 0\}$  be a sequence of positive numbers, satisfying condition (4). Then

$$(18) \quad \sup_k \frac{M_{|n_k|}}{M_{\langle n_k \rangle}} = \infty.$$

Under condition (18) there exists a sequence  $\{\alpha_k : k \geq 0\} \subset \{n_k : k \geq 0\}$  such that  $\alpha_0 \geq 3$  and

$$(19) \quad \sum_{\eta=0}^{\infty} \frac{M_{\langle \alpha_k \rangle}^{(1-2p)/2}}{M_{|\alpha_k|}^{(1-2p)/2}} < c < \infty.$$

Let

$$f^{(n)} = \sum_{\{k; |\alpha_k| < n\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{\lambda M_{\langle \alpha_k \rangle}^{(1/p-2)/2}}{M_{|\alpha_k|}^{(1/p-2)/2}}$$

and

$$a_k = \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda} \left( D_{M_{|\alpha_k|+1}} - D_{M_{|\alpha_k|}} \right).$$

By applying Lemma 1 we can conclude that  $f \in H_p$ .

It is easy to show that

$$(20) \quad \widehat{f}(j) = \begin{cases} M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2}, \\ \text{if } j \in \{M_{|\alpha_k|}, \dots, M_{|\alpha_k|+1} - 1\}, \quad k = 0, 1, 2, \dots, \\ 0, \\ \text{if } j \notin \bigcup_{k=0}^{\infty} \{M_{|\alpha_k|}, \dots, M_{|\alpha_k|+1} - 1\}. \end{cases}$$

Moreover,

$$\sigma_{\alpha_k} f = \frac{1}{\alpha_k} \sum_{j=1}^{M_{|\alpha_k|}} S_j f + \frac{1}{\alpha_k} \sum_{j=M_{|\alpha_k|+1}}^{\alpha_k} S_j f := I + II.$$

Let  $M_{|\alpha_k|} < j \leq \alpha_k$ . Then, by applying (20), we get that

$$(21) \quad S_j f = S_{M_{|\alpha_k|}} f + M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2} \left( D_j - D_{M_{|\alpha_k|}} \right).$$

By using (21) we can write  $II$  as

$$\begin{aligned} II &= \frac{\alpha_k - M_{|\alpha_k|}}{\alpha_k} S_{M_{|\alpha_k|}} f + \frac{M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2}}{\alpha_k} \sum_{j=M_{|\alpha_k|+1}}^{\alpha_k} \left( D_j - D_{M_{|\alpha_k|}} \right) \\ &: = II_1 + II_2. \end{aligned}$$

It is easy to show that

$$\|II_1\|_{weak-L_p}^p \leq \left( \frac{\alpha_k - M_{|\alpha_k|}}{\alpha_k} \right)^p \left\| S_{M_{|\alpha_k|}} f \right\|_{weak-L_p}^p \leq c_p \|f\|_{H_p}^p < \infty.$$

By using part a) of Theorem 1 for the estimation of  $I$  we have that

$$\|I\|_{weak-L_p}^p = \left( \frac{M_{|\alpha_k|}}{\alpha_k} \right)^p \left\| \sigma_{M_{|\alpha_k|}} f \right\|_{weak-L_p}^p \leq c_p \|f\|_{H_p}^p < \infty.$$

Let  $x \in I_{\langle \alpha_k \rangle - 1, \langle \alpha_k \rangle}^{\langle \alpha_k \rangle + 1}$ . Under condition (4) we can conclude that  $\langle \alpha_k \rangle \neq |\alpha_k|$  and  $\langle \alpha_k - M_{|\alpha_k|} \rangle = \langle \alpha_k \rangle$ . If we apply equality (8) for  $l = 1$  and estimate (12) in Lemma 6 for  $II_2$  we obtain that

$$\begin{aligned} |II_2| &= \frac{M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2}}{\alpha_k} \left| \sum_{j=1}^{\alpha_k - M_{|\alpha_k|}} \left( D_{j+M_{|\alpha_k|}} - D_{M_{|\alpha_k|}} \right) \right| \\ &= \frac{M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2}}{\alpha_k} \left| \psi_{M_{|\alpha_k|}} \sum_{j=1}^{\alpha_k - M_{|\alpha_k|}} D_j \right| \\ &\geq c_p M_{|\alpha_k|}^{1/2p-1} M_{\langle \alpha_k \rangle}^{(1/p-2)/2} (\alpha_k - M_{|\alpha_k|}) \left| K_{\alpha_k - M_{|\alpha_k|}} \right| \\ &\geq c_p M_{|\alpha_k|}^{1/2p-1} M_{\langle \alpha_k \rangle}^{(1/p+2)/2}. \end{aligned}$$

It follows that

$$\begin{aligned} &\|II_2\|_{weak-L_p}^p \\ &\geq c_p \left( M_{|\alpha_k|}^{(1/p-2)/2} M_{\langle \alpha_k \rangle}^{(1/p+2)/2} \right)^p \mu \left\{ x \in G_m : |IV_2| \geq c_p M_{|\alpha_k|}^{(1/p-2)/2} M_{\langle \alpha_k \rangle}^{(1/p+2)/2} \right\} \\ &\geq c_p M_{|\alpha_k|}^{1/2-p} M_{\langle \alpha_k \rangle}^{1/2+p} \mu \left\{ I_{\langle \alpha_k \rangle - 1, \langle \alpha_k \rangle}^{\langle \alpha_k \rangle + 1} \right\} \geq \frac{c_p M_{|\alpha_k|}^{1/2-p}}{M_{\langle \alpha_k \rangle}^{1/2-p}}. \end{aligned}$$

Hence, for large  $k$ ,

$$\begin{aligned} &\left\| \sigma_{\alpha_k} f \right\|_{weak-L_p}^p \\ &\geq \|II_2\|_{weak-L_p}^p - \|II_1\|_{weak-L_p}^p - \|I\|_{weak-L_p}^p \\ &\geq \frac{1}{2} \|II_2\|_{weak-L_p}^p \geq \frac{c_p M_{|\alpha_k|}^{1/2-p}}{2 M_{\langle \alpha_k \rangle}^{1/2-p}} \rightarrow \infty, \text{ as } k \rightarrow \infty. \end{aligned}$$

The proof is complete.  $\square$

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